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Saltation of uniform grains in air

By P. R. OWEN
Department of Aeronautics, Imperial College, London

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The interaction between a turbulent wind and the motion of uniform saltating grains of sand or soil, so massive as to fail to enter into suspension, is examined on the basis of two complementary hypotheses. The first asserts that the effect of the moving grains on the fluid outside the region to which saltation is confined is similar to that of solid roughness of height comparable with the depth of the saltation layer. The second requires the concentration of particles engaging in the saltation to adjust itself so that the shear stress exerted by the wind on the ground —different from that acting on the fluid outside the saltation layer by an amount accountable to the change in horizontal momentum suffered by the particles in their passage through the fluid—is just sufficient to maintain the sand-strewn surface in a mobile state.

Existing experimental data on the wind profiles outside the saltation region and the horizontal flux of particles through it are shown to be consistent with these hypotheses.

The second hypothesis implies a self-balancing mechanism for controlling the concentration of saltating particles. For if the concentration is too low the shear stress at the surface rises above the value required merely to secure mobility and more particles are encouraged to leave the surface; conversely, too large a concentration depresses the surface stress, and the consequent loss of surface mobility inhibits saltation and reduces the concentration of particles until equilibrium is restored.

1. Introduction

Wind blowing over soil or a sand-strewn surface will, if the particles are heavy enough and the windspeed is not too large, induce a motion known as 'saltation' in which individual grains ejected from the surface follow distinctive trajectories under the influence of air resistance and gravity. They fail to enter into suspension, as they would if the particles were very fine or the wind violent: instead, once lifted from the surface, they rise a certain distance, travel with the wind and then descend, either to rebound on striking the surface or to embed themselves in it and eject other particles.

The approximate domain of grain size and windspeed—or, more precisely, friction velocity in the flow away from the surface and beyond the region containing the moving grains—in which saltation of quartz-like material can occur is illustrated in figure 2.

A vivid and detailed account of the phenomenon is given by Bagnold in his book *The Physics of Blown Sands and Desert Dunes* (1941); indeed, it is largely to this delightful work that I owe my interest in the subject. But there are two questions which the book imperfectly answers. What is the effect of the saltation on the airflow at large distances from the surface? What determines the concentration of particles engaging in the saltation?

The second question is a crucial one, not only to the problem of saltation but of particle erosion in general, and reduces to a demand for knowledge of the concentration of particles adjacent to the surface. If this knowledge were available for all cases of grain erosion, one could, in principle, work out the amount of fine dust carried from the ground into suspension in the atmosphere; or its hydraulic (and more difficult†) counterpart, the quantity of silt suspended in a river. However, no attempt will be made here to enter into a discussion of these latter problems; we shall confine our attention to the more primitive case of pure saltation in the atmosphere or other gaseous medium.

In order to try to answer the specific questions posed above, two simple, interacting hypotheses will be introduced: but it would be as well to set out first of all those aspects of the phenomenon that will be taken for granted and the highly simplified model which will be employed to represent them.

In the first place, the particles are assumed to be uniform in size and shape and nearly spherical, an allowance if necessary for any departure from sphericity being made by introducing a shape factor into the calculation of their air resistance. Secondly, the entire particulate motion, which in reality must be endowed with a certain randomness, is regarded as repetitive such that the trajectory shape of one particle is identical with that of any other and is independent of time and of distance along the surface, as sketched in figure 1; thus, the region to which saltation is confined—it will be called the *saltation layer*—can be allotted a definite height. Thirdly, the motion is treated as two-dimensional or, at any rate, homogeneous with respect to a direction transverse to the wind. Fourthly, appeal will be made to the copious observations of Bagnold (1941) and Chepil (1945*a*) in assuming that more often than not the initial part of a trajectory is nearly vertical. Finally, it is supposed that the saltating grains are so massive that the air drag they experience is small compared to their weight.

Evidently, as orderly a model of the saltation process as this conceals a number of problems; for instance, the way the saltation is initiated and the precise mechanics of impact between the grains and the surface. Wherever possible, we shall avoid these delicate matters and restrict discussion to the two questions posed previously: the behaviour of the flow beyond the saltation layer and the concentration of particles within it.

The hypotheses to be put forward are as follows. Hypothesis (i): *the saltation layer behaves, so far as the flow outside it is concerned, as an aerodynamic roughness whose height is proportional to the thickness of the layer.* Hypothesis (ii): *the concentration of particles within the saltation layer is governed by the condition that the shearing stress borne by the fluid falls, as the surface is approached, to a value just sufficient to ensure that the surface grains are in a mobile state.*

The argument underlying the first hypothesis is simply that the turbulence

† Because hydrodynamic forces arising from the shear in the fluid are important when the densities of the particulate material and the fluid are comparable (see §2).

within the saltation layer, and communicated to the outside flow, is generated primarily by the movement of the particles; for, in their motion relative to the fluid, they shed wakes which introduce irregularities into the fluid flow. Viewed over the whole saltation layer, and including contributions from upgoing particles and downgoing particles, the broad scale of the irregularities, interpreted as turbulence, must be similar in order of magnitude to the thickness of

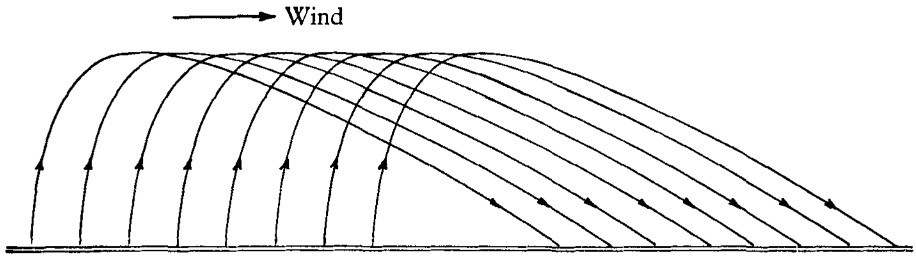

FIGURE 1. Trajectories of the particles.

the layer. (Here we acknowledge the randomness in the spatial distribution of particles which our model and especially figure 1, disregarded.) Since the characteristic ascensional velocity of the particles in the layer is u_τ, its thickness must be comparable with $u_\tau^2/2g$, where u_τ is the friction velocity in the outer, particle-free flow. It follows, on analogy with the aerodynamic behaviour of solid roughness, that the velocity profile in the fluid outside the saltation layer should obey the law

$$U/u_\tau = 2{\cdot}5 \log (2gy/u_\tau^2) + D' \quad (y \geqslant h),$$

where U is the windspeed at height y above the surface, h is the thickness of the layer and D' is a constant. There are abundant data to construct a test of the above relation; such a test is made in figure 3 where U/u_τ is plotted against $\log_{10}(2gy/u_\tau^2)$. The points in that figure, which covers a wide range of particle sizes and wind-speeds and includes both uniform sands and non-uniform soils, fall about a line whose slope is 5·75 (i.e. $2{\cdot}5 \log_e 10$). It therefore appears that the hypothesis is acceptable.

The second hypothesis is more subtle. Over most of its trajectory a particle is urged along by the wind, so that an elementary volume of fluid enclosing the particle at any instant is subjected to a reactive force which tends to retard its motion. Since it has been assumed that conditions are invariant with respect to distance along the surface, the force on the fluid element must be balanced by a gradient in the shearing stress acting on its horizontal surfaces such that the stress decreases as the ground is approached. Another way of putting the argument, bearing in mind that downgoing particles have had time to acquire a larger horizontal component of velocity than upgoing particles, is that the closer one gets to the surface the greater is the amount of horizontal momentum transported vertically by the particles; since the total rate of momentum transfer, or shear stress, must be constant throughout the saltation layer, the proportion carried by the fluid has to fall. Clearly, the greater the concentration of particles, the smaller is the residual shear stress transmitted to the ground as skin friction. Whether the skin-friction is manifested by a genuine viscous stress, as in the case of an aerodynamically smooth surface, or a form drag acting on the particles composing a rough or rippled surface is immaterial because the function of the aerodynamic force in securing mobility is to counteract the weight of a particle and its moment about points of contact with neighbouring particles, as well as cohesive forces, all of which are largely independent of the topography of the surface. Nor do the detailed contours of the surface have a profound effect on the flow in the saltation layer, for, consistent with the hypotheses, both the mean flow and the turbulence within the layer are controlled by the motion of the particles: the one by the drag acting on them and the other by the wakes they shed.

The skin friction, τ_0, required to initiate movement among otherwise stationary grains was first measured by Shields (1936) who found from experiments with sand grains in water that $\tau_0/\sigma g d = \beta$ (a measure of the ratio of the hydrodynamic force on a surface particle to its weight), where d is the diameter of a grain, σ is its density and β is a slowly varying function of Reynolds number $u_\tau/d\nu$ and of order 10^{-2}.† Subsequently, Bagnold (1937) and Chepil (1945*b*), repeating the experiments in air, discovered β to be somewhat smaller than Shields's value, yet still of order 10^{-2}. However, the values of β appropriate here are not identical with those required to *initiate* grain movement, although they may be expected to be comparable in order of magnitude because, in addition to the mean aerodynamic force exerted on a particle lying in the surface, turbulent fluctuations in both pressure and skin friction are present and may be expected to depend on whether or not saltation is in progress. The difference between the two situations was recognized by Bagnold (1941) who determined β for grain movement to be first detectable on an initially passive surface—what he termed the 'fluid threshold'—and then the value of β for which saltation, artificially induced, would just be maintained—the 'impact threshold'. He found β at the fluid threshold to be 0·01 and at the impact threshold to be 0·0064, results which were subsequently confirmed by Chepil (1945*b*). Henceforth we shall suppose that the shear stress required near the surface to sustain an equilibrium saltating flow is $0{\cdot}0064\,\sigma g d$.

In ascribing the condition for mobility to purely aerodynamic forces it might be objected that we have ignored the contribution to the force on the surface from the bombardment by particles and accordingly have rejected the possibility that the total shear stress at the surface might be relevant. Our view here is that the role played by the total stress is to impart a horizontal drift to the extreme layers of particles, described as a 'surface creep'; more particularly, the stress component due to impacting particles is communicated successively to adjoining particles lying in or near the surface by a sequence of more or less horizontal impulses and cannot by itself much affect a particle's preparedness to leave the surface vertically.

The dependence of conditions at the surface on τ_0 can be extended still further by arguing that if τ_0 falls below $\beta\sigma g d$ mobility is lost and a particle striking the surface will not readily dislodge another, having to expend energy in a mechanical

† Shields's graph of β against $u_\tau d/\nu$ is reproduced in several books; for example, those of Prandtl (1952), Rouse (1950) and Leliavsky (1955).

process of loosening the surface grains which would otherwise have been accomplished by the aerodynamic forces; conversely, a value of τ_0 larger than $\beta\sigma g d$ secures too great a mobility, including the projection into the fluid of particles purely under the action of aerodynamic forces. These observations lead to the postulate of a self-balancing mechanism for controlling the concentration of particles in the saltation layer. For, suppose the concentration exceeded the level required to make the skin friction equal to $\beta\sigma g d$; then, with a given value of the shear stress in the outer flow, the shear stress at the surface would fall below $\beta\sigma g d$ and the loss in surface mobility would inhibit the saltation and reduce

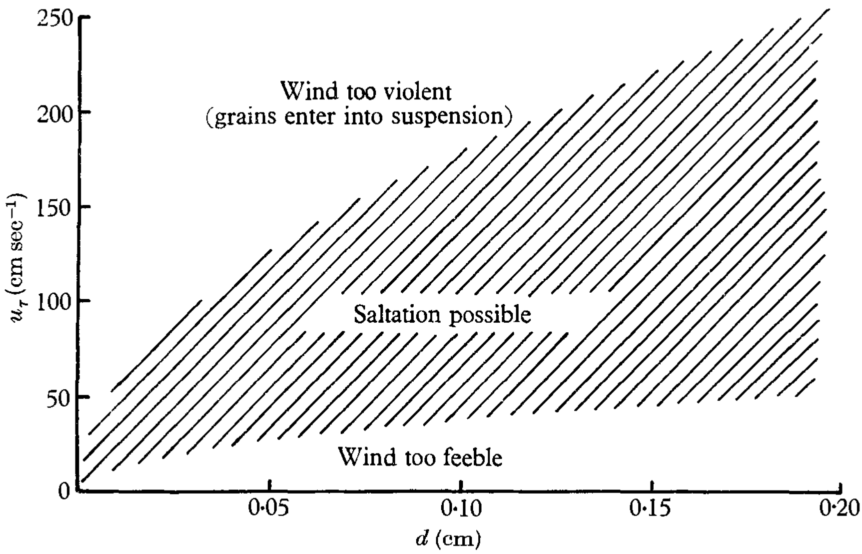

FIGURE 2. Range of windspeed and grain size in which saltation of quartz grains can occur in the atmosphere; u_τ is the friction velocity in the flow outside the saltation layer.

the concentration until equilibrium was restored. Similarly, a defect in concentration would raise the skin friction above $\beta\sigma g d$ and thereby encourage a greater number of particles to be drawn from the surface.

The test of hypothesis (ii) can only be made indirectly because there are no known measurements of the concentration of grains in a saltating airflow. On the other hand, there are many observations of the total streamwise flux through the saltation layer, an especially detailed set being that of Zingg (1953). It turns out that the measurements of the mass flux are consistent with a theory based on the hypothesis and supplemented by an energy argument which relates the vertical velocity of a particle leaving the surface to the ratio of the drag it experiences in flight to its weight.

Finally, it may be observed that saltation can be prescribed by

$$O(10^{-2}) < \rho u_\tau^2/\sigma g d < O(1),$$

where the lower limit is required for surface mobility and the upper by the condition that the grains do not enter into suspension. The consequent range of u_τ and d for the occurrence of the saltation of quartz grains in the atmosphere is roughly defined in figure 2.

2. Motion of the particles

A surface $y = 0$ of sand or soil is imagined to be swept by a turbulent wind blowing in the x-direction. The friction velocity of the wind, u_τ, in relation to the size of the grains, is supposed to be sufficiently large to erode the surface, but not large enough to cause the grains to enter into suspension. The resulting motion is one of saltation in which the velocity components (u, v) of a typical particle of mass m at a distance y from the ground are u_1, v_1 in the upgoing part of the trajectory and u_2, v_2 in the downgoing part.

The equations of the particle motion, assuming that drag is the only significant aerodynamic force and that interference between neighbouring (sparsely distributed) particles is negligible, are

$$m\ddot{y} + mg + R(s)\dot{y}/s = 0, \tag{1}$$

$$m\ddot{x} - R(s)\,(U - \dot{x})/s = 0, \tag{2}$$

where $s = \{\dot{y}^2 + (U - \dot{x})\}^{\frac{1}{2}}$, $U(y)$, to be determined in §4, is the wind velocity, taken to be everywhere parallel to the ground, and R is the drag.

The simple form into which equations (1) and (2) have been cast is a consequence of expressing the aerodynamic force in terms solely of the relative velocity between the particle and the local fluid. However, the fluid in the saltation layer is necessarily sheared and a lift force ensues from the interaction between the motion of the particle and the environmental vorticity.† The latter, in order of magnitude, is u_τ/h; hence, the lift on a grain travelling relative to the fluid with a velocity $O(u_\tau)$ is $O(\rho u_\tau^2 d^3/h)$, and its ratio to the weight of a grain is $O(\rho u_\tau^2/\sigma g h)$: that is, $O(\rho/\sigma)$. For sand grains in air, ρ/σ is $O(10^{-3})$ and the lift may therefore be neglected in comparison with the drag which, in terms of the particle weight, is $O(\rho u_\tau^2/\sigma g d)$: according to the inequality stated at the end of the previous section, the latter quantity must be greater, in order of magnitude, than 10^{-2} for saltation to occur. On the other hand, for sand in water ρ/σ is comparable with unity and the lift force plays a significant part in determining the motion of a particle.

If the sand is of such a diameter that the drag experienced by any grain is small compared with its weight (but not so small as the lift), the vertical motion of a grain is largely gravity-controlled and $R(s)$ in (1) can be expressed in an approximate form. The most convenient approximation is one in which the drag is taken to depend linearly on the relative velocity between the particle and the fluid,

$$R(s) = Ks. \tag{3}$$

(3) is, of course exact in the Stokes regime of flow about a particle but, in general, the Reynolds number of a saltating particle, defined as $u_\tau d/\nu$, is substantially larger than unity and typically lies in the range 1 to 50, where the drag follows more nearly a $\frac{3}{2}$-power of the relative velocity. The horizontal component of the particle motion is, on the other hand, resistance-controlled and consideration has to be given to the choice of K in order that an acceptable approximation to R is achieved in (2). Since it will appear later, §7, that the wind velocity in the

† A lift force, similar in order of magnitude, can also result from the spin of a particle.

saltation layer takes a nearly uniform value of Du_τ, it is appropriate to select K so that the drag is given *correctly* by the linear relation when the relative velocity between a particle and the fluid is *equal* to Du_τ. Accordingly, K will be defined by

$$KDu_\tau = C_D(Du_\tau d/\nu)\,\tfrac{1}{2}\rho D^2 u_\tau^2 \tfrac{1}{4}\pi d^2, \tag{4}$$

where C_D is found from experimental values for spheres as quoted, for example, by Schlichting (1955). Since sand or soil particles are never truly spherical, the value of d in (4) should be multiplied by a shape factor, but in the range of Reynolds numbers concerned here the shape factor lies near to unity and may be ignored. Anticipating §3, D may be set equal to 10; its precise magnitude is unimportant because, in the relevant range of $u_\tau d/\nu$ the opposing effects of a change in D on the relative velocity and on C_D render K fairly insensitive to the value of D.

In preference to using K itself, we define a velocity v_0 by

$$Kv_0 = mg. \tag{5}$$

Then

$$\frac{u_\tau}{u_0} = \tfrac{15}{2} C_D\left(10\frac{u_\tau d}{\nu}\right)\frac{\rho u_\tau^2}{\sigma g d}. \tag{6}$$

u_τ/v_0 may be regarded as a rough measure of the ratio of the drag on a particle moving with a velocity of order u_τ relative to the fluid to its weight;† it would be accurately so if the Reynolds number $10u_\tau d/\nu$ were less than unity, when v_0 would become the true terminal velocity of the particle.

Equations (1) and (2) can now be written

$$\ddot{y} + g + g\dot{y}/v_0 = 0, \tag{7}$$

$$\ddot{x} - g(U - \dot{x})/v_0 = 0. \tag{8}$$

In order to settle the initial conditions on $\dot{x}$ and $\dot{y}$ we shall invoke the observation made by Bagnold (1941) and Chepil (1945*a*) that particles freshly ejected from the ground mostly travel vertically upwards. Bagnold also deduced from measurements of the length of their bound along the surface that their initial velocity was comparable with u_τ. Although governed in detail by a complex mechanical process at the surface, it might be remarked that the tendency for the initial velocity of a particle to be vertical is not inconsistent with our assumption, hypothesis (ii), of a mobile and elastically anisotropic granular surface, as a consequence of which the horizontal component of a particle's momentum is able to be absorbed on impact whilst its vertical component is reversed. Other, less vague, situations giving rise to an initial vertical velocity can be imagined. For example, a layer of surface particles, uniform except for one that is displaced above the layer and rests on its neighbours will, when struck horizontally, eject the anomalously placed grain vertically. It might also be noted that since the height attained by a particle is proportional to the *square* of the vertical component of its velocity, for a given initial speed only those particles which leave the surface nearly vertically are able to survive an extensive trajectory and therefore can be said to participate in the saltation.

† The drag associated with the apparent mass of an *accelerating* particle, in terms of its weight can be shown to be $O(10\rho u_\tau/\sigma v_0)$ which is negligibly small.

With the conditions at time $t = 0$, $x = y = \dot{x} = 0$ and $\dot{y} = \alpha u_\tau$, where α may confidently be expected to be of order unity (it will be shown in § 8 to be a function of u_τ/v_0), the solutions of (7) and (8) are

$$y = \frac{v_0^2}{g}\left\{\epsilon - \frac{v}{v_0} + \log\left[\frac{1+v/v_0}{1+\epsilon}\right]\right\}, \tag{9}$$

$$u_1 = g\,(1+v_1/v_0)\int_0^y \frac{U\,dy}{v_1(v_1+v_0)}, \tag{10}$$

$$u_2 = g(1+v_2/v_0)\left\{\int_0^h \frac{U\,dy}{v_1(v_1+v_0)} - \int_y^h \frac{U\,dy}{v_2(v_2+v_0)}\right\}. \tag{11}$$

$\epsilon = \alpha u_\tau/v_0$ is assumed to be appreciably less than unity.

The height h of the saltation layer and the vertical velocities of the particles within it are found from (7) and (9) to be given by

$$(2gh)^{\frac{1}{2}} = \alpha u_\tau(1-\tfrac{1}{3}\epsilon+\ldots), \tag{12}$$

$$v_1 = \alpha u_\tau(1-y/h)^{\frac{1}{2}}\{1-\tfrac{1}{3}\epsilon+\tfrac{1}{3}\epsilon(1-y/h)^{\frac{1}{2}}+\ldots\}, \tag{13}$$

$$v_2 = -\alpha u_\tau(1-y/h)^{\frac{1}{2}}\{1-\tfrac{1}{3}\epsilon-\tfrac{1}{3}\epsilon(1-y/h)^{\frac{1}{2}}+\ldots\}. \tag{14}$$

3. Wind velocity profile outside the saltation layer

According to hypothesis (i) the saltation layer acts on the flow outside it, feeding it with turbulence, like a solid roughness of height comparable with h, and in the region $y \geqslant h$ the velocity profile can be expected to be represented by

$$U/u_\tau = 2{\cdot}5\log(2gy/u_\tau^2) + D'. \tag{15}$$

Measurements of $U(y)$ have been presented by Bagnold (1936), Chepil (1945*c*) and Zingg (1953), those of the latter author being particularly detailed and extensive, covering sand grain sizes from 0·015 to 0·084 cm and values of u_τ from about 40 cm sec^{-1} to 100 cm sec^{-1}; Chepil's experiments were made with natural soils. The velocity profiles were analysed to yield u_τ and then replotted against $\log_{10}(2gy/u_\tau^2)$ as shown in figure 3. The points cluster about the straight line

$$U/u_\tau = 2{\cdot}5\log(2gy/u_\tau^2) + 9{\cdot}7,$$

except when $2gy/u_\tau^2$ is appreciably less than unity, corresponding to the interior of the saltation layer.

The scatter in figure 3 is no larger than would be expected in view of the experimental difficulty of making velocity measurements in an unsteady, occasionally sand-laden flow by means of conventional pitot tubes. But not all the scatter can be thus summarily attributed to experimental inaccuracy (some part must be due to the author's inaccuracy in reading Zingg's and Chepil's data from the small-scale graphs given in their papers), for on detailed inspection of the orignal data the points were found to fall, with some system, into groups according to particle diameter or u_τ/v_0, suggesting that the constant D' in (15) is dependent on u_τ/v_0. Nonetheless, the dependence, even if genuine, is a weak one and any attempt to include it in the analysis would not be at all worthwhile. (It will appear subsequently that, in any case, a prediction of the particle flux does not involve an accurate knowledge of D'.)

In terms of the actual height of the saltation layer,

$$U/u_\tau = 2{\cdot}5 \log (y/h) + D, \tag{16}$$

where
$$D = D' + 5 \log \alpha - 5\epsilon/3.$$

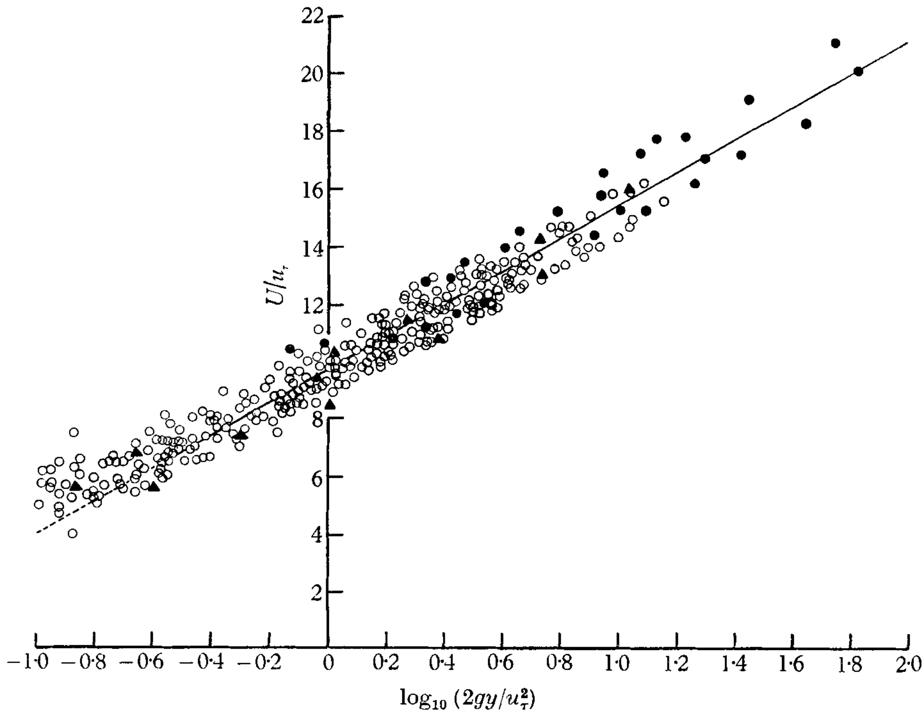

FIGURE 3. Wind-velocity profile outside the saltation layer. Uniform sand: ○, Zingg (1953); ▲, Bagnold (1941); non-uniform soil: ●, Chepil (1945*a*, *b*, *c*).

Since α is comparable with unity and ϵ is not large, D cannot be widely different from D'; however, its precise value is dependent on u_τ/v_0 (more strongly so than the dependence of D' suggested by the analysis of figure 3), owing to the presence of the terms in α and ϵ.

Equation (15) enables an interpretation of Bagnold's 'focus' to be made. Bagnold (1941) observed that his measured velocity profiles for different values of u_τ seemed to converge on a small region—not clearly enough defined to be called a point—in the neighbourhood of 3 mm from the ground, which he described as a focus. The behaviour evidently arises from the tendency for the change in u_τ to be balanced by a change in h in its effect on U at a certain height above the ground. Thus, from (15), $\partial U/\partial u_\tau = 0$ when $2{\cdot}5 \log (2gy) = 5 \log u_\tau - 4{\cdot}7$. With $u_\tau \approx 60\,\text{cm}\,\text{sec}^{-1}$, a rough average of the range of values of u_τ over which Bagnold's observations were made, it appears that U has a stationary value at $y \approx 2{\cdot}9$ mm.†

† Clearly, Bagnold's statement (1941) '...no matter how strongly the wind is made to blow...the wind velocity at a height of about 3 mm remains almost the same' cannot be true in general.

4. Wind profile in the saltation layer

The concentration at any point, the number of particles per unit volume, is $\psi(y)$, and is made up from a concentration $\psi_1(y)$ of upgoing particles and a concentration $\psi_2(y)$ of downgoing particles. Continuity of particle flux requires

$$v_1\psi_1 = \alpha u_\tau \psi_1(0),$$
$$v_2\psi_2 = v_2(0)\,\psi_2(0);$$

and, since the numbers leaving and arriving at the ground are equal,

$$\alpha u_\tau \psi_1(0) = -v_2(0)\,\psi_2(0).$$

It follows that the total concentration is

$$\psi = \alpha u_\tau \psi_1(0)\,[1/v_1 - 1/v_2]. \tag{17}$$

In order to preserve a flow that is steady on the average and independent of distance along the stream, conservation of momentum in x-direction† requires that

$$d\tau/dy = m\alpha u_\tau \psi_1(0)\, d(u_1 - u_2)/dy. \tag{18}$$

u_1 and u_2 follow from (10) and (11) with the wind velocity on $y = h$ set equal to Du_τ, as in (16):

$$u_1 = \epsilon D u_\tau \{f_1 - \epsilon[\tfrac{1}{3}f_1 - (1-y/h)^{\frac{1}{2}} f_1 + \tfrac{4}{3}f_2] + O(\epsilon^2)\}, \tag{19}$$

$$u_2 = \epsilon D u_\tau \{2F_1 - f_1 - \epsilon[\tfrac{2}{3}F_1 - \tfrac{1}{3}f_1 + \tfrac{4}{3}f_2 + (2F_1 - f_1)(1-y/h)^{\frac{1}{2}}] + O(\epsilon^2)\}, \tag{20}$$

where

$$f_1(\xi) = \int_\xi^1 (U/Du_\tau)\, d\xi, \quad f_2(\xi) = \int_\xi^1 (U/Du_\tau)\,\xi\, d\xi \quad \text{and} \quad F_1 = f_1(0); \quad \xi = (1-y/h)^{\frac{1}{2}}.$$

With the aid of (19) and (20), (18) can be written

$$d\tau/dy = 2mgD\psi_1(0)\,\epsilon(1+\tfrac{1}{3}\epsilon)\,(U/Du_\tau - \epsilon F_1)/\alpha(1-y/h)^{\frac{1}{2}} + O(\epsilon^3). \tag{21}$$

It will now be assumed that τ can be related to the velocity gradient dU/dy by an eddy viscosity which is constant over most of the saltation layer and equal to $\lambda u_\tau h$. The asumption of a constant eddy viscosity is the simplest that can be made in the absence of any detailed knowledge about the structure of the turbulence in the saltation layer, but it is not an implausible assumption because the presence of the grains provides a powerful mechanism of mixing. More specifically it may be noted that the vertical component of the turbulent intensity can be taken roughly as proportional to $v\psi$, since the turbulence is imagined to arise from the velocity defect in the wakes of the ascending and descending particles averaged over unit area parallel to the ground; but, in view of the condition of continuity of particle flux, $v\psi$ is constant throughout the layer. Hence, a uniform

† The corresponding equation of motion in the y-direction shows that there must be a pressure gradient,

$$dp/dy = -m\alpha u_\tau \psi_1(0)\, d(v_1 + v_2)/dy,$$

needed to balance the vertical flux of momentum carried by the grains and thereby inhibit any bulk vertical motion of the fluid. It can be shown that the buoyancy force on a grain, associated with this pressure gradient, is very small compared with the drag.

vertical intensity of the turbulence coupled with a uniform scale over (most of) the saltation layer suggests that the assumption of a constant eddy viscosity may not be wildly unrealistic.

The above argument naturally breaks down in the immediate vicinity of the surface, especially in regard to the scale of the turbulence, for there the eddies are constrained by the presence of the ground. In any case, viscous stresses provide an increasing contribution to τ as $y \to 0$. But the fraction of the saltation layer occupied by fluid under the direct influences of the surface and viscosity is small, $O[(1/\beta)(\rho/\sigma)(\nu/u_\tau d)]$, and can be ignored, provided that no attempt is made to impose the no-slip condition on the solution of (21).

Writing $y/h = \eta$, (21) becomes

$$\frac{d^2(U/Du_\tau)}{d\eta^2} = \tfrac{9}{16}\gamma^2 \frac{(U/Du_\tau - \epsilon F_1)}{(1-\eta)^{\frac{1}{2}}}, \tag{22}$$

where

$$\gamma^2 = \frac{16}{9}\frac{m\alpha\psi_1(0)}{\rho\lambda}\epsilon(1-\tfrac{1}{3}\epsilon). \tag{23}$$

No conjecture can be offered at this stage about the order of magnitude of γ^2; although proportional to ϵ, it contains the factor $m\psi_1(0)/\rho$, the concentration of particles by weight at the surface, whose magnitude is unknown. The question will be reconsidered in §7.

The solution of (22) is

$$U/Du_\tau = \epsilon F_1 + (1-\eta)^{\frac{1}{2}}\{A\mathrm{I}_{\frac{2}{3}}(z) + B\mathrm{I}_{-\frac{2}{3}}(z)\}, \tag{24}$$

with $z = \gamma(1-\eta)^{\frac{3}{4}}$. $\mathrm{I}_{\frac{2}{3}}(z)$ and $\mathrm{I}_{-\frac{2}{3}}(z)$ are Bessel functions of imaginary argument.

The boundary conditions (24) has to satisfy are as follows:

$$U/Du_\tau = 1 \quad \text{on} \quad \eta = 1, \quad \text{hypothesis (i)}; \tag{25}$$

$$\left[\frac{d(U/Du_\tau)}{d\eta}\right]_{\eta=1} = \frac{1}{\lambda D}, \quad \text{continuity of shear stress}; \tag{26}$$

$$\left[\frac{d(U/Du_\tau)}{d\eta}\right]_{\eta\to 0} = \frac{1}{\lambda D}\frac{\beta\sigma g d}{\rho u_\tau^2}, \quad \text{hypothesis (ii)}. \tag{27}$$

Writing $\beta\sigma g d/\rho u_\tau^2 = \phi$, the constants A and B in (24) are found from (25), (26) and (27) to be

$$A = -a(\gamma,\phi)(1-\epsilon F_1),$$

where

$$a(\gamma,\phi) = 2.3^{-\frac{1}{2}}\pi(\tfrac{1}{2}\gamma)\mathrm{I}_{\frac{1}{3}}(\gamma)/[(\tfrac{1}{2}\gamma)^{\frac{1}{3}}\mathrm{I}_{-\frac{1}{3}}(\gamma)\Gamma(\tfrac{2}{3})-\phi]; \quad B = b(\gamma)(1-\epsilon F_1), \tag{28}$$

and

$$b(\gamma) = (\tfrac{1}{2}\gamma)^{\frac{2}{3}}\Gamma(\tfrac{1}{3}), \tag{29}$$

together with the relation determining γ,

$$1/\lambda D = \tfrac{3}{2}(\tfrac{1}{2}\gamma)^{\frac{2}{3}}a(\gamma,\phi)(1-\epsilon F_1)/\Gamma(\tfrac{2}{3}). \tag{30}$$

The solution of (30) will be deferred to §7.

5. Mass flux of particles

The mass flux of particles across unit width of a plane perpendicular to the fluid stream is

$$G = \int_0^h m(\psi_1 u_1 + \psi_2 u_2)\,dy$$
$$= m\alpha u_\tau \psi_1(0)\int_0^h (u_1/v_1 - u_2/v_2)\,dy. \tag{31}$$

Using (13), (14), (19) and (20) we find that

$$G = \tfrac{9}{8}(\rho u_\tau^2/g)\,\alpha\gamma^2\lambda D\left\{F_1 - \epsilon\left[\tfrac{2}{3}F_1 + \frac{8}{3}\int_0^1 f_2\,d\xi - \frac{4}{3}\int_0^1 f_1\xi\,d\xi\right] + O(\epsilon^2)\right\}.$$

Recalling that

$$f_1(\xi) = \int_\xi (U/Du_\tau)\,d\xi \quad \text{and} \quad f_2(\xi) = \int_\xi^1 (U/Du_\tau)\,\xi\,d\xi,$$

we may easily show that

$$\int_0^1 f_2\,d\xi = 2\int_0^1 f_1\xi\,d\xi.$$

Hence $$Gg/\rho u_\tau^3 = \tfrac{9}{8}\alpha\gamma^2\lambda D\left\{F_1 - \epsilon\left[\tfrac{2}{3}F_1 + 4\int_0^1 f_1\xi\,d\xi\right] + O(\epsilon^2)\right\}. \tag{32}$$

6. Evaluation of f_1 and F_1

The functions f_1 and F_1, together with $\int_0^1 f_1\xi\,d\xi$, can be found with the aid of (24); all the integrations can be performed analytically, using the well-known relation

$$\int z^\nu \mathrm{I}_{\nu-1}(z)\,dz = z^\nu \mathrm{I}_\nu(z).$$

Thus, we obtain

$$f_1(\xi) = \epsilon F_1(1-\xi) + \tfrac{2}{3}\gamma^{-1}(1-\epsilon F_1)\{a(\gamma,\psi)\,[\mathrm{I}_{-\frac{1}{3}}(\gamma) - \xi^{\frac{1}{2}}\mathrm{I}_{\frac{1}{3}}(\gamma\xi^{\frac{3}{2}})]$$
$$+ b(\gamma)\,[\mathrm{I}_{\frac{1}{3}}(\gamma) - \xi^{\frac{1}{2}}\mathrm{I}_{\frac{1}{3}}(\gamma\xi^{\frac{3}{2}})]\}. \tag{33}$$

Hence, $$F_1 = c(\gamma,\phi) + \epsilon c(\gamma,\phi)\,[1 - c(\gamma,\phi)] + O(\epsilon^2),$$

where $$c(\gamma,\phi) = \tfrac{2}{3}\gamma^{-1}\{a(\gamma,\phi)\,[(\tfrac{1}{2}\gamma)^{-\frac{1}{3}}/\Gamma(\tfrac{2}{3}) - \mathrm{I}_{-\frac{1}{3}}(\gamma)] + b(\gamma)\,\mathrm{I}_{\frac{1}{3}}(\gamma)\}. \tag{34}$$

Also $$\int_0^1 f_1\xi\,d\xi = -\tfrac{1}{3}\gamma^{-1}a(\gamma,\phi)\,[\mathrm{I}_{-\frac{1}{3}}(\gamma) - \tfrac{4}{3}\gamma^{-1}\mathrm{I}_{\frac{2}{3}}(\gamma)]$$
$$+ \tfrac{1}{3}\gamma^{-1}b(\gamma)\,[\mathrm{I}_{\frac{1}{3}}(\gamma) - \tfrac{4}{3}\gamma^{-1}\mathrm{I}_{-\frac{2}{3}}(\gamma) + \tfrac{4}{3}\gamma^{-1}(\tfrac{1}{2}\gamma)^{-\frac{2}{3}}/\Gamma(\tfrac{1}{3})] + O(\epsilon). \tag{35}$$

7. The constants λ and D: solution for small γ

The question raised in §4 concerning the magnitude of γ can be resolved with the aid of (30), provided that a rough estimate of λD is available. For this purpose we note from figure 3 that, since α is $O(1)$, D is of order 10. Moreover, the substance of hypothesis (i) is that the turbulence in the outer flow near $y = h$ is generated within the saltation layer; accordingly, it may be supposed that the eddy viscosity is continuous across $y = h$. But it has been assumed, and verified in figure 3, that the eddy viscosity in $y > h$ is $0{\cdot}4u_\tau y$, so that, for continuity across $y = h$, λ must be equal 0·4.

We are therefore led to infer that λD is about 4. With λD of this approximate magnitude, (30) can be satisfied only if γ^2 is small, in which case the Bessel functions can be represented by their expansions about $\gamma = 0$,

$$\mathrm{I}_\nu(\gamma) = \frac{(\frac{1}{2}\gamma)^\nu}{\Gamma(1+\nu)}\left[1+\frac{(\frac{1}{2}\gamma)^2}{1+\nu}+\ldots\right].$$

A physical interpretation of the conclusion that γ is small is suggested by (22). The effect of the grains on the velocity distribution of the fluid in the saltation layer is to tend to render it uniform in a way similar to the action of a gauze on an inhomogeneous stream, by slowing down those parts of the stream which move faster, and therefore suffer the greater resistance, and by speeding up the more slowly moving parts. In this, it is aided by the vigorous turbulent mixing within the layer. As a result of the evening-up process,† the derivative d^2U/dy^2 can be expected to be small, and (22) indicates that this is so if γ^2 is small.

For small γ, it is found that

$$a(\gamma,\phi) = 3\,\frac{(\frac{1}{2}\gamma)^{\frac{4}{3}}\,\Gamma(\frac{2}{3})}{(1-\phi)}\left[1-\frac{1}{6}\frac{1+\phi}{\lambda D(1-\epsilon)}+\ldots\right], \tag{36}$$

$$c(\gamma,\phi) = 1-\tfrac{3}{4}(\tfrac{1}{2}\gamma)^2(1+\phi)/(1-\phi)+\ldots, \tag{37}$$

$$\int_0^1 f_1\xi\,d\xi = \tfrac{1}{6}+\tfrac{1}{4}(\tfrac{1}{2}\gamma)^2+\frac{9}{20}\frac{(\frac{1}{2}\gamma)^2}{1-\phi}\left[1-\frac{1}{6}\frac{1+\phi}{\lambda D(1-\epsilon)}\right]+\ldots, \tag{38}$$

and

$$\frac{1}{\lambda D} = \frac{9}{2}\frac{(\frac{1}{2}\gamma)^2}{1-\phi}\left[1-\epsilon+\tfrac{3}{4}(\tfrac{1}{2}\gamma)^2\frac{(1+\phi)(2\epsilon-1)}{1-\phi}+\ldots\right]. \tag{39}$$

Hence,

$$(\tfrac{1}{2}\gamma)^2 = \frac{2}{9}\frac{1-\phi}{\lambda D(1-\epsilon)}\left[1+\frac{1}{6}\frac{1+\phi}{\lambda D}+\ldots\right]. \tag{40}$$

It follows from (32) that

$$\frac{Gg}{\rho u_\tau^3} = \alpha(1-\phi)\left[1-\frac{1}{6}\frac{1+\phi}{\lambda D}+O\left(\frac{\epsilon}{\lambda D}\right)\right].$$

The term $\frac{1}{6}(1+\phi)/\lambda D$ is small enough in general to be neglected, and we are left with the simple result

$$Gg/\rho u_\tau^3 = \alpha(1-\phi), \tag{41}$$

which involves neither λ nor D.

The fluid velocity in the saltation layer is given by

$$U/Du_\tau = 1-(1/\lambda D)(1-\eta)+\tfrac{3}{5}(1/\lambda D)(1-\phi)(1-\eta)^{\frac{5}{3}}+O(1/\lambda^2D^2). \tag{42}$$

8. The quantity α: comparison with experimental data

It is clear from (41) that the magnitude of α plays a crucial part in determining the flux of particles through the saltation layer: evident on physical grounds as a result of the thickening of the saltation layer, achieved by an increase in α,

† The departure of the points in figure 3 from the straight line, at the bottom left of the figure corresponding to values of y within the saltation layer, is not inconsistent with the suggestion that the velocity tends to become uniform.

outweighing the thinning-out of the concentration of particles due to their more vigorous upward motion.

There is no reason to believe that α is a constant; but we should not expect its value to be predictable by aerodynamic argument alone unless some special tendency exists for the mechanics of an impact between a particle and the surface to be adjusted to the condition that the surface acts as a receptacle for the excess kinetic energy a particle possesses just before impact. Precisely such a tendency is implied by hypothesis (ii), in which it is supposed that the mobile state of the surface enables the kinetic energy received by a particle from the fluid through which it travelled to be absorbed in a surface creep. As a consequence, the dependence of α on u_τ/v_0 can be revealed by constructing the energy balance within the *fluid* occupying the saltation layer, without regard to the details of the impacts with the surface.

Consider a box of fluid in the saltation layer, of unit width and breadth, extending from the top of the layer to the edge of the viscous region adjacent to the surface. Work is done on the fluid in the box by the aerodynamic stresses acting on its horizontal surfaces. Of this work, part goes into accelerating the particles in a horizontal direction, part into turbulent energy through the action of the shearing stresses on the mean flow and the remainder directly into turbulent energy from the wakes of the particles. More formally, the energy equation can be arrived at through multiplying (18) by U and integrating over the saltation layer:

$$\int_0^h U\frac{d\tau}{dy}dy = m\alpha u_\tau \psi_1(0)\int_0^h \frac{Ud(u_1-u_2)}{dy}dy$$
$$= m\alpha u_\tau \psi_1(0)\left\{\int_0^h (U-u_1)\frac{du_1dy}{dy} - \int_0^h (U-u_2)\frac{du_2dy}{dy} + \tfrac{1}{2}[u_2(0)]^2\right\}. \quad (43)$$

Substituting for $(U-u_1)$ and $(U-u_2)$ from (8) and for $m\alpha\psi_1(0)$ from (23), and casting into a non-dimensional form, (43) can be written

$$1-\frac{\phi U(0)}{Du_\tau} = \lambda D\int_0^1\left[\frac{d}{d\eta}(U/Du_\tau)\right]^2 d\eta$$
$$+\frac{9}{16}\frac{\gamma^2}{\epsilon(1-\frac{1}{3}\epsilon)}\left\{\tfrac{1}{2}\epsilon(1-\tfrac{2}{3}\epsilon)\left[\int_0^1\frac{(U/Du_\tau-u_1/Du_\tau)^2}{v_1/\alpha u_\tau}d\eta\right.\right.$$
$$\left.\left.-\int_0^1\frac{(U/Du_\tau-u_2/Du_\tau)^2}{v_2/\alpha u_\tau}d\eta\right]+\tfrac{1}{2}[u_2(0)/Du_\tau]^2\right\}. \quad (44)$$

The left-hand side represents the rate of working by the external airborne stresses; the terms on the right arise respectively from the rate of conversion of mean flow energy to turbulent energy, the rate of work done against the drag of the particles (equal to the rate of appearance of turbulent energy derived from their wakes and its ultimate dissipation) and the flux of particulate kinetic energy out of the bottom of the box.

All the terms in (44) can be evaluated with the assistance of the approximate expressions given at the end of § 7. (44) is then found to be satisfied if

$$\epsilon = \text{const.} = \tfrac{1}{3}+O(\epsilon^3, 1/\lambda^2D^2). \quad (45)$$

It follows that
$$\alpha \propto (v_0/u_\tau). \tag{46}$$

The simple relation between α and v_0/u_τ afforded by (46) can be easily examined in the light of available experimental data on G. According to (41)
$$Gg/\rho u_\tau^3(1-\phi) = \alpha,$$
so that a plot of $Gg/\rho u_\tau^3(1-\phi)$ against v_0/u_τ should yield a straight line. Figure 4 shows such a plot and, indeed, the data points fall about the line
$$Gg/\rho u_\tau^3(1-\phi) = 0{\cdot}25 + \tfrac{1}{3}v_0/u_\tau. \tag{47}$$

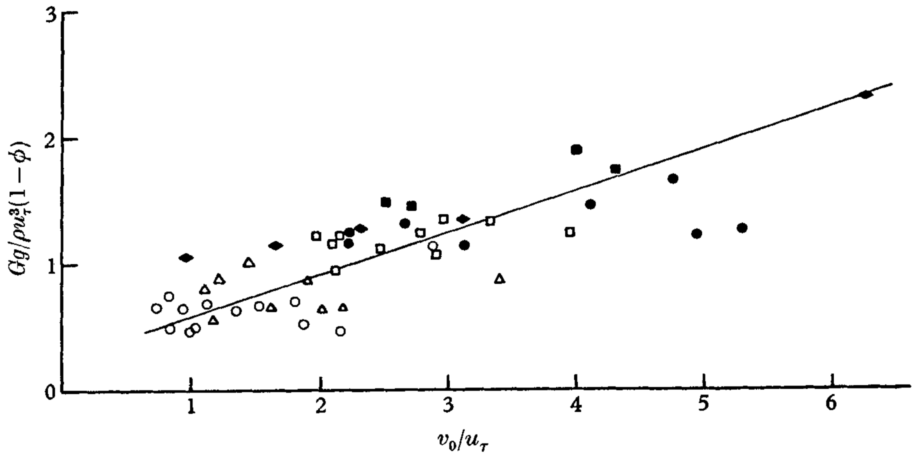

FIGURE 4. The relation between α and v_0/u_τ. Zingg (1953), particle diameter (cm): ○, 0·02; △, 0·0275; □, 0·036; ●, 0·0505; ■, 0·0715. Bagnold (1941), particle diameter (cm): ◆, 0·025.

Hence, for large v_0/u_τ to which the entire analysis is restricted, $\alpha \sim \frac{1}{3}v_0/u_\tau$, in agreement with (45). Since the solution to (44) is only accurate to $O(1/\lambda D)$ and the slope of the line in figure 4 depends slightly on the value of D used to define v_0/u_τ no great significance can be attached to the apparently *exact* agreement with (45).

It may be remarked that figure 4 provides a rather delicate test of (46)—since what is displayed is the comparatively gentle dependence of G on α that remains after the strong variable u_τ^3 has been absorbed in the ordinate—and the scatter of the experimental points is considered to be by no means excessive.

With (47) written in full, and the restriction to large v_0/u_τ removed by appeal to figure 4, it appears that the mass flux of uniform grains saltating in a gaseous stream is given by
$$\frac{Gg}{\rho u_\tau^3} = \left[0{\cdot}25 + 0{\cdot}044\frac{\sigma g d/\rho u_\tau^2}{C_D(10u_\tau d/\nu)}\right]\left[1 - 0{\cdot}0064\frac{\sigma g d}{\rho u_\tau^2}\right]. \tag{48}$$

8.1. *Surface impact by fine grains*

Although the analysis is strictly valid only if u_τ/v_0 is small, the appearance of the constant term in (47) suggests that the naïve description, contained in our model, of what happens to a particle on impact with the surface requires modification when the grains are fine (large u_τ/v_0). For, according to the model, a particle striking the surface either bounces back into the fluid or ejects a fresh particle, the tangential momentum and excess kinetic energy of the incident particle being absorbed in a surface creep. But a light particle is not readily able to communicate its kinetic energy to, in effect, a large mass in the surface and we are led to speculate that such a particle rids itself of kinetic energy not only by shifting particles lying *in* the surface but by setting into motion other particles which then rise slightly *above* the surface, as sketched in figure 5. This conjecture simply requires the surface creep, in its function as an energy sink, to be extended to include grains which travel through the adjacent strata of fluid, and are thereby slowed-down.

Evidently, the effect of the symbiotic particles moving near the surface is to restore to the fluid some of the energy imparted by it to the higher-flying, saltating particles and can be roughly accounted for in (44) by an additional term on the

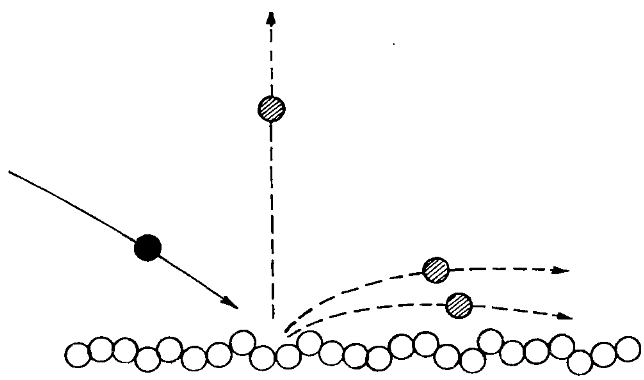

FIGURE 5. Conjectural impact of a fine particle with the surface: ●, incident particle; ◍, reflected and ejected particles.

left-hand side. In order to achieve consistency with (47), the additional term should be proportional to u_τ/v_0, which is plausible on physical grounds: because the need to invoke fluid-borne particles as absorbers of the excess kinetic energy possessed by an incident particle must increase as the mass of that particle decreases.

8.2. *Bagnold's law of particle flux*

Bagnold (1941) reported that his experimental values of G fitted a law of the form, $Gg/\rho u_\tau^3 \propto d^{\frac{1}{2}}$. It is of interest to see whether such a law is consistent with (48).† For this purpose, using (48), $\log_{10}(Gg/\rho u_\tau^3)$ is plotted against $\log_{10} d$ in figure 6 for a variety of values of u_τ.

The curves u_τ = const. in figure 6 are far from straight, but the parts corresponding to either small d or small u_τ can be approximated to by lines whose gradients are equal to $\frac{1}{2}$, in agreement with Bagnold's law. For larger values of

† It cannot be claimed that Bagnold's law is bound to be recovered in view of the fact that his data were used partly to establish (47) from figure 4. All Bagnold's points in that figure relate to a single particle diameter.

d or u_τ, however, the slope of the curves is nearer $\frac{3}{4}$ and it is relevant to note that Zingg (1953), whose experiments covered a wider range of particle sizes than those of Bagnold, observed that his results fitted a $d^{\frac{3}{4}}$-law.

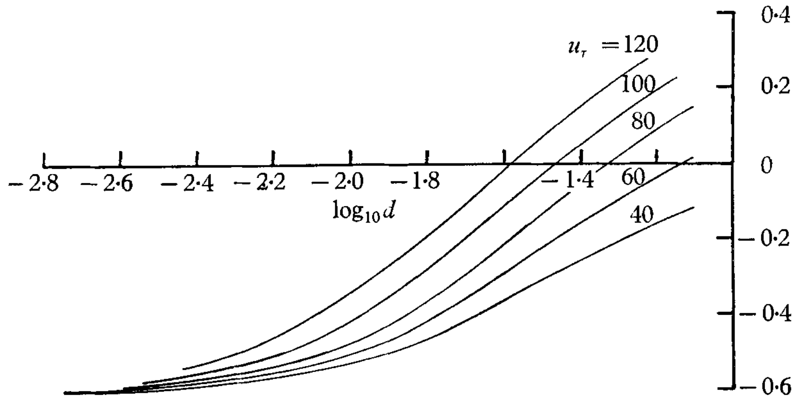

FIGURE 6. Test of Bagnold's law for quartz grains in air; d measured in cm, u_τ in cm/sec.

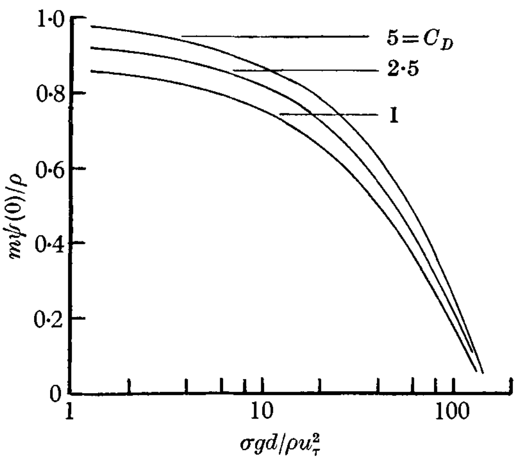

FIGURE 7. The surface concentration of particles.

8.3. *Particle concentration near the surface*

The question of the concentration of the particles in the neighbourhood of the surface, to which emphasis was given in the Introduction, can now be reconsidered. From (23) and (40),

$$m\psi_1(0)/\rho = (1-\phi)/2D\epsilon(1-\epsilon)(1-\tfrac{1}{3}\epsilon),$$

where, according to (16) and figure 3,

$$D = 9{\cdot}7 + 5\log\alpha - \tfrac{5}{3}\epsilon.$$

Also, making use of (13), (14) and (17), the total concentration is

$$\psi(0) = 2\psi_1(0)/(1-\tfrac{1}{3}\epsilon).$$

Hence

$$m\psi(0)/\rho = (1-\phi)/D\epsilon(1-\epsilon)(1-\tfrac{1}{3}\epsilon)^2. \qquad (49)$$

$m\psi(0)/\rho$, the concentration of particles by weight near the surface, is shown as a function of $\sigma gd/\rho u_\tau^2$ for a number of values of C_D in figure 7, from which it will be observed that over most of the range of $\sigma gd/\rho u_\tau^2$ the surface density of particles is comparable with that of the fluid. The concentration by *volume*, on the other hand, is dilute, of order 10^{-3}.

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